6. OZIRANER A.S., On the asymptotic stability and instability of motion relative to part of the variables. PMM, Vol.37, No.4, 1973.
7. RUMYANTSEV V.V., On the asymptotic stability and instability of motion relative to part of the variables. PMM, Vol.35, No.1, 1971.
8. ANDREYEV A.S., On the asymptotic stability and instability relative to part of the variables. Dokl. AN UzbSSR, No.5, 1982.
9. HATVANI L., On the partial asymptotic stability by Lyapunov function with semidefinite derivative. MTA, Számitáztechn. és automatiz. kut. intéz közl., No.26, 1982.
10. ROUCHE N. and PEIFFER K., Le théorem de Lagrange-Dirichlet et la deuxième méthode de Liapunoff. Ann. Soc. Sci., Bruxelles, ser. 1, Vol.81, No.1, 1967.
11. RUMYANTSEV V.V., Some problems of the stability of motion relative to part of the variables. In: Mechanics of a Continuous Medium and Kindred Problems of Analysis. Moscow, Nauka, 1972.
12. PEIFFER K and ROUCHE N., Lyapunov's second method applied to partial stability. J. Mec., Vol.8, No.2, 1968.
13. RUMYANTSEV V.V., On the control of orientation and stabilization of a satellite by rotors at points of libration. Publ. Inst. Mat., Nov. Sect., Vol.17, 1974.

Translated by J.J.D.

PMM U.S.S.R., Vol.48, No.5, pp. 514-522, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

THE USE OF LYAPUNOV'S SECOND METHOD TO ESTIMATE REGIONS OF STABILITY AND ATTRACTION*

V.G VERETENNIKOV and V.V. ZAITSEV

A definition of the stability region is given by extending the properties of Lyapunov's definition of sets of sizable measure. Constructive theorems on estimates of regions of stability and attraction are obtained by using certain developments of Lyapunov's second method for a wide class of autonomous and non-autonomous systems that satisfy both the Lipschitz and discontinuous conditions. The usual requirements imposed on the functions used in investigations of the stability region are somewhat reduced. For example, the requirement that the functions and their derivatives should have fixed sign are omitted.

1. Consider the equations of perturbed motion of the form

$$x^{\cdot} = f(x, t), \quad x \in R^n \tag{1.1 – 1.4}$$

By system (1.1) we mean an autonomous system, whose right side is $f(x)$, whose vector function $f(x)$ is such that the solution of the Cauchy problem in the region considered exists, is unique, and is continuous with respect to the initial conditions, excluding any arbitrarily small neighbourhood of singular points. For system (1.2) $f = f(x) \in C(R^n)$ and by Peano's theorem the integral curves can be continued to the boundary of any compact set, possibly in a non-unique way. In system (1.3) the single-valued vector function $f = f(x)$ is piecewise continuous. Among systems (1.3) with discontinuous single-valued right sides only those are considered for which each integral curve may be uniquely continued in the neighbourhood of any surface of discontinuity, and the number of such surfaces is finite. The vector function $f = f(x, t)$ in system (1.4) is such that the solutions retain the properties of the solutions of system (1.1) mentioned above.

The basic concepts and notation correspond to those used in /1/. In addition we shall introduce the upper right Dini derivative /2, 3/ denoted by D^+V; the connected subset F of the semiaxis $[t_0, \infty)$ such that $F = [t_0, T] \vee [t_0, \infty)$ $(T = \mathrm{const})$ (when investigating the properties of attraction $F = [t_0, \infty)$), $F_d{}^V = \{x \mid V(x) = d\}$; $H_{c(t)}^V = \{x \mid x = y(t, t_0, x_0) \wedge x_0 \in H_{c(t_0)}^V = H_{c_0}^V = \{x \mid V(x) \leqslant c_0\}\}$, $c_0 = \mathrm{const}$, and the integral curve $y(t, t_0, x_0)$ of the system considered under initial conditions x_0, t_0.

Let us assume that for the Lyapunov function $V \in C^1$ the following conditions are satisfied:

---

*Prikl.Matem.Mekhan., 48, 5, 714-724, 1984

$$(\exists c)\ (\forall x \in H_c^V)\ V^{\cdot}(x) \leqslant 0 \qquad (1.5)$$

$$(\forall x \in R^n \setminus \theta)\ V(x) > 0, \quad V(\theta) = 0 \qquad (1.6)$$

It is most convenient to evaluate the stability region as the set H_c^V. However, when conditions (1.5) and (1.6) are satisfied and the Lyapunov theorem on stability holds, it follows only that some stability region, not necessarily the same as H_c^V, exists.

For instance, a system of the form (1.1) may exist, which on some surface $F_{c_1}^V$ has singular points. It is possible to construct for such system a Lyapunov function V for which the set $F_{c_1}^V$ is the level surface. The behaviour of this function is similar to that shown in Fig.1, where $z = \sum_{i=1}^{n} x_i^2$. In that case $(\forall x \in F_{c_1}^V)\ V^{\cdot}(x) = 0$ and system (1.1) may be such that $(\forall x \in R^n \setminus \theta \setminus F_{c_1}^V)\ V^{\cdot}(x) < 0$. For example, when $n = 2$ for the system

$$x^{\cdot} = -g(z)\,x,\ y^{\cdot} = -g(z)\,y,\ z = x^2 + y^2$$

with the function $g = g(z)$ whose behaviour is defined by the curve in Fig.2 (a possible one is the function $g = -z^2 + zr_0$), when the Lyapunov function is $V = e^{-az}(1 - e^{-az})$, where $a = \ln 2/r_0$, $r_0 = \arg_z [V(z) = c_1]$, the region $H_{c_1}^V\ (c > c_1)$ is not a stability region. This situation also arises when the surface $F_{c_1}^V$ is a limiting cycle of system (1.1).

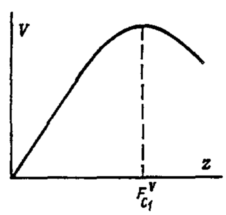

Fig.1

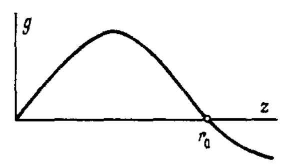

Fig.2

Even if the Lyapunov function $V(x)$ satisfies the condition

$$(\exists c > 0)\ (\forall x \in H_c^V \setminus \theta)\ V^{\cdot}(x) < 0$$

the set H_c^V may not only not be the region of attraction but also a region of stability /4/. In this case the measure mes_{R^n} of the set, where the right side of the system $f(x) \notin C^\infty$, is zero.

Similarly for a system of the form (1.3), when the condition

$$(\forall x \in H_c^V \setminus \theta)\ D^+ V(x) < 0 \qquad (1.7)$$

is satisfied, the set H_c^V is not necessarily a stability region. It was shown in /5, 6/ that for systems of the form (1.4) the set H_c^V when condition (1.7) is satisfied, is not necessarily a region of attraction.

To estimate the regions of stability and attraction we introduce special Lyapunov functions.

Definition 1. We shall call the function $V(x)$ that, on the set $G \in K$, satisfies conditions (2.2), 1°—4° and 6°—8° of /1/ and, also, condition (1.6) stated above, a Lyapunov-type function.

Definition 2. We shall call a function that is of the Lyapunov type on any subset $G_1 \in K$ in the region of definition $V(x)$, a strictly Lyapunov-type function.

We shall also consider Lyapunov-type functions (and strictly Lyapunov-type functions) in a wider class of sets. If the Lyapunov-type function $V(x)$ is defined on each of the sets $G_i\ (i = 1, 2, \ldots)\ G_i \in K$ such that $\lim G_i = G$ as $i \to \infty$, then $V(x)$ on G is defined as a function obtained by continuous continuation on the sequence $\{G_i\}$.

The function $V(x)$, shown in Fig.3, is not of the strictly Lyapunov type, since it is not a Lyapunov-type function on the set G_1.

Condition (1.6) is not obligatory for Lyapunov-type functions $V(x)$. It is sufficient to stipulate that the function $V(x)$ should have its absolute minimum at the point θ.

For functions of the strictly Lyapunov type, condition (1.6) may be replaced by the requirement that

$$\theta = \arg \min_{x \in D_V} V(x) = \arg \operatorname{loc} \min_{x \in D_V} V(x) = \arg \operatorname{abs} \operatorname{loc} \min_{x \in D_V} V(x)$$

where D_V is the region of definition of the function V, and θ is the point of the unique local minimum that coincides with the global minimum.

Note that according to the theory of convex functions /7/, properties (2.2) 4° and 8° in /1/ for functions of the strictly Lyapunov type are satisfied, if the functions are strictly convex or strictly quasiconvex.

It follows from Definition 1 that functions of the Lyapunov type may be defined by level surfaces and an arbitrary strictly monotonic function $R^{+1} \to R^{+1}$ defined on an arbitrary curve that intersects the level surfaces only at a single point.

The level surfaces of the Lyapunov-type functions are the boundaries of regions, and for various values of $V(x)$ satisfy the following properties (the Γ properties): they do not intersect, or touch each other, they contract to a point θ, are of measure (mes_{R^n}), equal to zero, and fill the whole set.

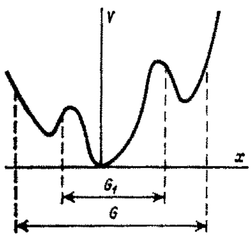

Fig.3

Any Lyapunov function is a Lyapunov-type function in any, possibly small, neighbourhood of θ. However the latter is not at all obligatory for the whole of the region of definition, although it is possible to separate among the Lyapunov functions a class of Lyapunov type functions, for example, positive definite quadratic forms that are functions of the strictly Lyapunov type.

The Lyapunov functions that satisfy the Krasovskii-Barbashin theorem on stability as a whole, are also Lyapunov-type functions (not necessarily strictly).

In investigations of stability as a whole the property of an infinitely large limit is not obligatory for Lyapunov-type functions.

The many Lyapunov functions used in the theory of stability /3—6, 8—21/ are not functions of the Lyapunov type in the whole region of definition. Thus Lyapunov functions that contain terms of the form

$$e^{\varphi(x)}, \sum_{i,j} a_{ij} (\cos mx)^i (\sin px)^j (m, p, a_{ij} = \text{const})$$

where $\varphi(x)$ is not a Lyapunov type function, as well as Lyapunov functions with various combinations of integrals of non-linearities are not necessarily functions of the Lyapunov type.

Lyapunov-type functions were used in /1/ to investigate the stability of compact sets. If, however, the set is not bounded, the level surfaces of the Lyapunov-type functions according to the construction algorithm, as defined by the lemma in /1/, are not closed. In that sense Lyapunov-type functions may not be Lyapunov functions that have closed level surfaces.

Later, we shall consider only functions of the strictly Lyapunov type, which for simplicity will be called functions of the Lyapunov type.

Lemma 1. Let Lyapunov type function $V(x) \in C^1$ exist. Then for the integral curves of system (1.1) the following statements hold:

from the condition

$$(\forall c, c_1 : c > c_1 > 0)\ (\forall x \in H_c^V \setminus H_{c_1}^V)\ V^\cdot(x) < 0$$

it follows that

$$(\forall x_0 \in H_c^V \setminus H_{c_1}^V) \Rightarrow \lim_{t \to \infty} x(t, t_0, x_0) \in H_{c_1}^V$$

from the condition

$$(\forall c, c_1 : c > c_1 > 0)\ (\forall x \in H_c^V \setminus \operatorname{int} H_{c_1}^V)\ V^\cdot(x) < 0$$

it follows that

$$(\forall x_0 \in H_c^V \setminus H_{c_1}^V) \Rightarrow (\exists T : t_0 \leqslant T < \infty)(\forall t \geqslant T)\, x(t, t_0, x_0) \in \operatorname{int} H_{c_1}^V$$

and from the condition

$$(\forall c, c_1 : c > c_1 > 0)\ (\forall x \in H_c^V \setminus H_{c_1}^V)\ V^\cdot(x) \leqslant 0$$

it follows that

$$(\forall c_0 : c_1 \leqslant c_0 \leqslant c)(\forall x_0 \in H_{c_0}^V) \Rightarrow (\forall t \in F)\, x(t, t_0, x_0) \in \operatorname{int}_1 H_{c_0}^V$$

The proof of the lemma is obvious.

The properties that follow from Lemma 1 are, as a rule, required to solve various applied problems.

Note that, when Lyapunov-type functions are used, problems of stability in-the-small are investigated by a single procedure with finite regions of stability, as a whole and also on sets /1/.

Let us now introduce the definition of the region of stability. It follows from the Lyapunov definition of stability that for stable solutions the property

$$(\exists \varepsilon_0)(\forall \varepsilon < \varepsilon_0)(\exists \delta > 0)(\forall x(t_0) \in \operatorname{int} S_\delta) \Rightarrow (\forall t \in F)\, x(t, t_0, x_0) \in \operatorname{int}_1 S_\varepsilon$$

is satisfied.

To solve systems (1.1) and (1.4) for any $\varepsilon > 0$ a δ_{max} exists which has such a property for a fixed ε. We select a fairly small $\varepsilon < \varepsilon_0$ and consider the one-to-one correspondence $S_\varepsilon \leftrightarrow S_{\delta_{max}}$. If in the respective region the system is stable, and if we continuously increase ε, the corresponding δ_{max} also increases continuously. The stability boundary is determined by the breakdown of the one-to-one correspondence $S_\varepsilon(t) \leftrightarrow S_{\delta_{max}}(t)$.

When this correspondence is satisfied, there exist in the stability region according to Nemytskii classification /19/, singular points of the type of centres, generalized centres, or centrofocal points.

The violation of correspondence in the ϑ neighbourhood of some surface, results in mutli-connectedness of the set of ω-limiting points for points from ϑ. Such multiconnectedness of the set of ω-limiting points for points of the set $G \in K$ does not impy that G is not a stability region.

Definition 3. We call the set of surfaces $\{\Omega\} = Q$ that cover the set $G \in K$, if set Q satisfies the properties Π on G.

Definition 4. We call the set $D_0(t)$: $(\forall t \in F)[D_0(t) \in D_1 \wedge D_0(t), D_1 \in K]$ the stability region, and call the system stable on D_0, if there exists a set of surfaces $Q(t)$ covering $D_0(t)$ such that the integral curves for $\forall t \in F$ do not emerge from $D_0(t)$ and, also,

$$\begin{gathered}(\forall t \in F)(\forall \beta \in Q(t))(\exists \psi \in Q(t))\, \mathrm{int}_1 \psi \subset \mathrm{int}_1 \beta \\ (\forall x_0 \in \mathrm{int}\, \psi) \Rightarrow (\forall \tau \geqslant t)\, x(\tau, t, x_0) \in \mathrm{int}_1 \beta \wedge \\ \{(\forall \beta_1, \beta_2 \in Q(t))\, \mathrm{int}_1 \beta_1 \subset \mathrm{int}_1 \beta_2 \wedge \mathrm{int}_1 \beta_1 \not\supset \mathrm{int}_1 \beta_2\} \Rightarrow \\ (\exists \psi_1, \psi_2 \in Q(t))(\forall \psi_3 \in Q(t))\, \mathrm{int}_1 \psi_1 \subset \mathrm{int}_1 \psi_3 \subset \mathrm{int}_1 \psi_2 \wedge \\ \mathrm{int}_1 \psi_2 \not\subset \mathrm{int}_1 \psi_3 (\exists x_0 \in \mathrm{int}\, \psi_3)(\exists \tau > t)\, x(\tau, t, x_0) \notin \mathrm{int}_1 \beta_1\end{gathered}$$

Note that in the Definition 4 instead of S_ε, S_δ a more general form of sets is used, namely, the surfaces β, ψ that are time independent; the set Q depends on time, since the region boundary may depend on time; the set D_1 is independent of time.

Note that when the system has a region of stability, the solution $x = \theta$ is Lyapunov stable.

Definition 5. We call D_0 the attraction region, if it is a stability region and the system has the property of attraction on D_0.

Lemma 2. Systems (1.1)–(1.4) are stable on the set D_0, if and only if the correspondence $\beta \leftrightarrow \psi_{max}$ is satisfied on the whole set D_0. Here β and ψ_{max} are the respective surfaces in Definition 4.

Proof of necessity. If the system is stable on D_0, then $(\forall \beta \in Q)(\exists \psi)$ is such that $(\forall \beta_1$: $\mathrm{int}_1 \beta_1 \subset \mathrm{int}_1 \beta \wedge \beta_1 \neq \beta)$ is the corresponding surface $\psi_1 \subset \mathrm{int}_1 \psi$. The integral curves that begin in $\mathrm{int}\, \psi_1$ do not reach β. On the other hand,

$$(\forall \beta_2 \in Q\colon \mathrm{int}_1 \beta_2 \supset \mathrm{int}_1 \beta \wedge \beta_2 \neq \beta)(\exists \psi_2)$$

is such that integral curves belonging in ψ_2 at some instant of time, and do not belong to the set $\mathrm{int}_1 \beta$, exist. Thus the correspondence $\beta \leftrightarrow \psi_{max}$ occurs.

The proof of sufficiency is obvious.

2. Let us consider a number of theorems on the evaluation of regions of stability and attraction.

Theorem 1. a) Let a Lyapunov-type function $V(x)$ exist such that the condition (1.5) $(V(x) \in C^1)$, required to solve systems (1.1)–(1.4), or the condition

$$(\forall x \in H_c^V)\, D^+ V(x) \leqslant 0 \tag{2.1}$$

is satisfied. Then H_c^V is the region of stability.

b) Let there be a Lyapunov-type function $V(x)$ such that for solving systems (1.1) and (1.2) condition (1.7) is satisfied, or when $V(x) \in C^1$ the condition

$$(\forall x \in H_c^V \setminus \theta)\, V^{\cdot}(x) < 0 \tag{2.2}$$

is satisfied. Then H_c^V is the region of attraction.

If for systems (1.3) condition (1.7) is satisfied, and in some neighbourhood $\vartheta(S)$ of any discontinuity surface, the inequality

$$(\forall x \in \vartheta(S))\, D^+ V(x) < -\beta < 0 \quad (\beta = \mathrm{const})$$

holds, then H_c^V is the region of attraction.

Proof. From condition (2.1) and the properties of Lyapunov-type functions it follows that the integral curves cannot intersect the level surfaces in the direction from inside to outside. This shows that the set H_c^V is a region of stability.

To prove the property of attraction we select an arbitrary point $x_0 \in H_c^V$ and consider the function $V(x(t, t_0, x_0)) = \Phi(t)$. By condition (1.7) $\Phi(t)$ $(t \geqslant t_0)$ is a decreasing function. By the theorem on the derivative of a monotonic function /3/ we have

$$\Phi(t) = \int_{t_0}^{t} \Phi^{\cdot}(\tau)\, d\tau - \eta(t) \tag{2.3}$$

where the function $\Phi^{\cdot}$ is defined almost everywhere and $\eta(t)$ is a decreasing function.

For any sequence $\{t_j\}$, where every t_j is taken from the region of definition of $\Phi^{\cdot}$ such that $\lim_{j\to\infty} t_j = \infty$, the condition

$$\underline{\lim}_{j\to\infty} D^+\Phi(t_j) = 0 \tag{2.4}$$

is satisfied.

Let us assume the opposite, i.e. that

$$\underline{\lim}_{j\to\infty} D^+\Phi(t_j) = -\alpha < 0 \quad (\alpha = \text{const})$$

Then the sequence $\{t_{ji}\}$: $\lim_{j_i\to\infty} t_{ji} = \infty$ exists for which $D^+\Phi(t_{ji}) < -\alpha + \varepsilon < 0$ $(\varepsilon = \text{const})$ for fairly large number j_i. Then also the function $\Phi^{\cdot}$ is such that $\Phi^{\cdot}(t_{ji}) < -\alpha + \varepsilon < 0$, and the inequality

$$\Phi^{\cdot}(t_{ji}) < -\alpha + \varepsilon + \varepsilon_1 < 0 \quad (\varepsilon_1 = \text{const})$$

holds almost everywhere in some neighbourhood of the point t_{ji}. Since the behaviour of the function on the set of zero measure does not affect the Lebesgue integral, the first term on the right side of (2.3) decreases without limit as t increases. Hence a finite time T: $T > t_0$ and $\Phi(T) < 0$ exist. But this is impossible. Consequently, for any sequence $\{t_j\}$ selected outside some set of zero measure (the Lebesgue measure) the condition (2.4) holds.

Condition (2.4) means that $x(t_j, t_0, x_0) \to \theta$ for solutions of systems (1.1) and (1.2) since $D^+V(x) = 0$ only at the point $x = \theta$.

The proof given above also holds for solutions of (1.3), since by the condition of the theorem there are no ω-limiting points on each of the surfaces of discontinuity.

Theorem 1 does not apply to systems of the form (1.2)–(1.4) for evaluating the stability regions, if instead of functions of the Lyapunov type we take Lyapunov functions to which Definitions 1 and 2 do not apply.

Theorem 2. Let there be a function of the Lyapunov type $V(x) \in C^1$ such that for solutions of systems (1.1) and (1.2) conditions (1.5) are satisfied and for some function of the Lyapunov-type $W(x) \in C^1$ the condition $W^{\cdot}(x) \neq 0$ $(\forall x \in M)$ is satisfied on the set $M = \{x \mid V^{\cdot}(x) = 0\}$. Then H_c^V is the region of attraction.

Proof. Let there be an ω-limiting set $\lambda^+(H_c^V)$ for points from H_c^V. It can be shown that by virtue of conditions (1.5) $\lambda^+(H_c^V)$ can only be a subset of M. The proof that the set M does not contain ω-limiting points can be obtained in the same way as the proof of the property of attraction in Theorem 1.

Theorem 2 is an extension of the theorem of Krasovskii-Barbashin /5, 8/ on systems (1.2) in the form used in /3/ for investigating the properties of weak attraction.

Theorem 3. Let Lyapunov-type functions $V(x) \in C^1$ and $V_1(x)$ such that for solutions of system (1.4) the condition

$$(\forall x \in H_c^V \setminus \theta) \quad 0 > -V_1(x) \geqslant \sup_{t \in F} V^{\cdot}(x) \tag{2.5}$$

is satisfied. Then H_c^V is a region of attraction.

Proof. According to Theorem 1 the system considered here is stable. Let us select an arbitrary $x_0 \in H_c^V$. By virtue of the properties of solutions of system (1.4) we have $V(x(t, t_0, x_0)) \in C_t(F)$.

Let us assume the contrary, i.e. that the integral curve $x(t, t_0, x_0)$ does not reach some level surface $F_{c_1}^V$ $(0 < c_1 \leqslant V(x_0) \leqslant c)$. Then

$$V(x(t, t_0, x_0)) - V(x_0) = \int_{t_0}^{t} V^{\cdot}(\tau, t_0, x_0)\, d\tau \tag{2.6}$$

We put $\max\limits_{x \in H_{c_1}^V} V_1(x) = a$ $(a > 0)$ and from (2.6) we obtain

$$V(x(t, t_0, x_0)) \leqslant -a(t - t_0) + V(x_0)$$

where the first term on the right side decreases without limit as t increases. Hence for an arbitrary $c_1 : c > c_1 > 0$ there is a finite time T at which all integral curves beginning in H_c^V become $H_{c_1}^V$. From this follows the property of attraction.

Theorem 3 holds when condition (1.5), or (2.1) is satisfied, and condition (2.5) is satisfied beginning at some $T < \infty$ $(\forall t \geqslant T)$. This theorem also holds if in condition (2.5) the upper right Dini's derivative is used.

Let us consider some corollaries of Theorem 3.

Corollary 1. Suppose Lyapunov-type functions $V(x)$ and $V_1(x)$ exist, such that condition (2.1) is satisfied, and there is a measurable set F_1 consisting of a finite or denumerable number of intervals for which

a) $(\exists M : \infty > M \geqslant 0)\ \mathrm{mes}_{R^1} F_1 < M$

b) $0 \geqslant -V_1(x) \geqslant \sup\limits_{F \setminus F_1} D^+V(x)$

Then H_c^V is a region of attraction for (1.4).

Corollary 2. If Lyapunov-type functions $V(x)$, $V_1(x)$, and $V_2(x)$ and sets F_1 and F_2 consisting of denumerable intervals exist, such that:

a) $F = F_1 \bigcup F_2$,

b) $(\forall i)\ (\forall c_i, d_i : [c_i, d_i] \subset F_1)\ (\exists a_i, b_i : [a_i, b_i] \subset F_2 \wedge a_i > d_i \wedge$
$b_i - a_i \geqslant d_i - c_i)\ [(c_{i+1} > c_i) \Rightarrow (a_{i+1} > a_i)] \wedge \{[c_{i+1}, d_{i+1}] \cap$
$[c_i, d_i] = \phi \Rightarrow [a_i, b_i] \cap [a_{i+1}, b_{i+1}] = \phi\} \wedge$,

c) $\wedge \{(\forall x \in H_c^V)\ 0 \geqslant -V_1(x) \geqslant \max\limits_{t \in [a_i, b_i]} D^+V(x)\} \wedge$

d) $\wedge \{(\forall x \in H_c^V \setminus \theta)\ \max\limits_{t \in [c_i, d_i]} D^+V(x) < V_2(x)\}$

e) $(\forall x \in H_c^V)\ V_1(x) \geqslant V_2(x)$,

and also

$$(\forall x \in H_c^V)\ (\forall t \in F)\ (\exists L\colon 0 \leqslant L < \infty)\ |f(x, t)| < L \tag{2.7}$$

then the set $H_d^V \subset H_c^V$ for which $(\forall t \in F)\ H^V_{c_0(t)} \subset H_c^V :\ c_0(t_0) = d$ is the region of attraction.

Corollaries 1 and 2 enable us to waive not only the requirement for the functions themselves to be of fixed sign, but also their derivatives.

Corollary 3. If the right sides of system (1.4) satisfy conditions (2.7) and there are Lyapunov-type functions $V(x), V_1(x) \in C^1$ and a continuous function $\eta(t)$ such that $(\forall t \in F)$ $(\forall x \in H_c^V)\ V^{\cdot}(x) \leqslant \eta(t) V_1(x)$ and, moreover, one of conditions:

a) $(\forall t \in F)\ \eta(t) \leqslant 0 \wedge \lim_{t\to\infty} \eta(t) < 0$,

b) $(\forall t \in F)\ |\eta(t)| < M < \infty \wedge (\forall t \in F \setminus F_1 : \mathrm{mes}_{R^1} F_1 < M_1 < \infty)\ \eta(t) < 0$,

c) $(\forall t \in F)\ |\eta(t)| < M < \infty \wedge \lim_{t\to\infty} \eta(t) = \eta_0 < 0$,

d) $(\exists \{T_i\} : \bigcup\limits_i [T_i, T_{i+1}] = F,\ i = 1, 2, \ldots, \wedge T_1 = t_0 \wedge (\forall i) T_{i+1} -$
$T_i \leqslant \tau < \infty) \wedge (\forall t \in F)\ |\eta(t)| < M < \infty \wedge (\exists q > 0)\ q < 2M\tau$
$(\exists a)\ (\forall x \in H_c^V)\ aV(x) > V_1(x)$,

and

$$(\forall t \in [T_i, T_{i+1}])\ (\forall x \in H_c^V)\ |V_1^{\cdot}(x)| \leqslant \frac{V_1(x) \ln k}{\tau} \wedge \int\limits_{T_i}^{T_{i+1}} \eta(t)\,dt \leqslant -q < 0 \wedge 1 < k \leqslant \frac{2M\tau}{2M\tau - q} \tag{2.8}$$

are satisfied, then for a) the set H_c^V is the region of attraction; for b) the region of attraction for system (1.4) is the set $H_{c_1}^V$, where $c_1 = c/\exp(aMM_1)$, $(\forall x \in H_c^V)\ aV(x) > V_1(x)$; for the cases of c) and d) the maximum set $H_{c_0}^V : (\forall t \in F)\ H^V_{c_0(t)} \subset H_c^V$ is also the region of attraction for system (1.4).

Proof. When condition a) is satisfied, the corollary follows from Theorems 1 and 2. When condition b) is satisfied, using the theorem of congruence /22/, the function $V(x(t, t_0, x_0))$ for solutions $x(t, t_0, x_0)$ contained in the set H_c^V cannot according to the norm increase in a time interval equal to M_1 more rapidly than the solutions of equations $y^{\cdot} = aMy$. Then

$$y = y(0) \exp [aM(t - t_0)] \leqslant y(0) \exp (aMM_1)$$

We conclude from this that the corollary also holds, when condition b) is satisfied.

The proof of the corollary when condition c) is satisfied follows from its validity when conditons a) and b) are satisfied.

To prove the validity of the corollary when condition d) is satisfied we shall prove that the following statements hold:

1) for any i on the segment $[T_i, T_{i+1}]$ $(\exists H_{c_1}^V)$

$$(\forall t \in [t_0, T_{i+1}])\; H^V_{c_1(t)} \subset H_c^V$$

2) there is a finite T such that not a single integral curve in $H_{c_1}^V$ in the time interval $[t_0, T]$ leaves H_c^V even when $(\forall t \geqslant T)$, and

3) the set $H_{c_1}^V$ cannot contain any ω-limiting points, other than $x = \theta$.

Let us prove statement 1). By the theorem of congurnece /22/ the function $V(x(t, t_0, x_0))$ for $x(t, t_0, x_0) \in H_c^V$ and $t \in [T_i, T_{i+1}]$ cannot increase more rapidly than the solution of the optimization problem

$$\sup_{\eta(t)} V(y(T_{i+1}))$$

where $y(T_{i+1})$ is the solution of the differential equation $y^{\cdot} = \eta(t)\, ay$ at the instant of time T_{i+1}, $y(T_{i+1}) = y(T_{i+1}, T_i, y_0)$; $y_0 = V(x(T_i, t_0, x_0))$ for $x_0 \in H_c^V$ and the function $\eta(t)$ satisfies the constraint (2.8).

Hence a finite L exists such that for $t \in [T_i, T_{i+1}]$ we have $V(x(t, t_0, x_0)) \leqslant V(x_0) \exp L$ and, consequently, statement 1) holds.

By virtue of constraint (2.8) we have in any time interval $[T_i, T_{i+1}]$

$$V(x(T_{i+1}, t_0, x_0)) - V(x(T_i, t_0, x_0)) = \int_{T_i}^{T_{i+1}} V^{\cdot}\, dt \leqslant$$

$$\int_{A_i^+} V_1(x)\,\eta(t)\,dt + \int_{A_i^-} V_1(x)\,\eta(t)\,dt \leqslant V_1(x(T_i, t_0, x_0))\,\eta_i^+ k +$$

$$V_1(x(T_i, t_0, x_0))\,\eta_i^-/k = V_1(x(T_i, t_0, x_0))\,(\eta_i^+ k + \eta_i^-/k) \leqslant$$

$$\frac{[-4Mq\tau(M\tau + \eta_i^-) + q^2\eta_i^-]\,V_1(x(T_i, t_0, x_0))}{2M\tau(2M\tau - q)} < 0$$

where

$$A_i^+ = \{t \mid t \in [T_i, T_{i+1}] \wedge \eta(t) \geqslant 0\}, \quad A_i^- = \{t \mid t \in [T_i, T_{i+1}] \wedge$$
$$\eta(t) < 0\}, \quad \eta_i^+ = \int_{A_i^+} \eta(t)\,dt, \quad \eta_i^- = \int_{A_i^-} \eta(t)\,dt$$

The existence of some T such that the whole set $H_{c_1}^V$ cannot emerge from H_c^V is evident from the inequality obtained. Hence statement 2) holds.

The proof of statement 3) follows from the satisfaction of condition (2.8). In fact, $(\forall i)$ on the segment $[T_i, T_{i+1}]$ when $T_i > T$ the system does not only not emerge from H_c^V, but for some interval of time $[t_i, t_{i+1}]$ the inequality $\eta(t) \leqslant 0$ for $(\forall t \in [t_i, t_{i+1}])$ is satisfied. Moreover that interval can be selected so that

$$V(x(T_i, t_0, x_0)) \leqslant V(x(t_i, t_0, x_0)) < V(x(T_{i+1}, t_0, x_0))$$

The further proof of statement 3) and its corollary is obvious.

3. *Examples*. 1°. Consider a free solid body subjected to the moment of external resistance forces. On the assumptions made in /20/ we write the Euler equations in the form

$$I_i \frac{d\omega_i}{dt} + (I_{i+2} - I_{i+1})\,\omega_{i+1}\omega_{i+2} = -\varkappa(t)\,|\omega|^{\alpha-1}\,\omega_i$$
$$(i = 1,\ 2,\ 3;\ i + 3 = i)$$

where I_i are the moments of inertia of the body relative to the principal central axes of inertia of the latter, and ω_i are the projections of angular velocity of the body on the same axes.

For the unperturbed motion $\omega_1 = \omega_2 = \omega_3 = 0$.

Consider a Lyapunov-type function of the form

$$V = \frac{1}{2}\sum_{i=1}^{3} I_i\omega_i^2; \quad V^{\cdot} = -\varkappa(t)\Big(\sum_{i=1}^{3}\omega_i^2\Big)^{(\alpha+1)/2}$$

If for $t \geqslant t_0$ the condition

$$(\exists\delta)(\forall t \in F)\ \varkappa(t) \geqslant \delta > 0 \tag{3.1}$$

is satisfied, the system as a whole is uniformly stable. If the inequality $(\forall t \in F)$ $\varkappa(t) \geqslant 0$, and $\varkappa(t) = 0$ on the set F_1 such that $\mathrm{mes}_{R^1} F_1 < M < \infty$, $M = \mathrm{const}$, the system is as a whole stable. When the inequality $|\varkappa(t)| < M < \infty$ is satisfied and outside some set F_0 condition (3.1) is satisfied and $\mathrm{mes}_{R^1} F_0 < M < \infty$, the system is also as a whole stable.

2°. Let us investigate the stability of motion in the vertical plane of a dynamically and geometrically symmetric body in a fluid. The equations of perturbed motion in the case considered here can be represented in the form /20/

$$(1 + k_1)\, V^{\cdot} = C_x{}^V V + C_x{}^\beta \beta + C_x{}^\alpha \alpha + C_x{}^\omega \omega \qquad (3.2)$$
$$(1 + k_2)\, \alpha^{\cdot} = C_y(\alpha) + C_y{}^\omega \omega + C_y{}^\beta \beta + C_y{}^V V$$
$$(1 + k_3)\, \rho_z\, \omega^{\cdot} = m_z(\alpha) + m_z{}^\omega \omega + m_z{}^\beta \beta, \quad \beta^{\cdot} = \omega$$

where V is the modulus of velocity, α is the angle of attack, β is the angle of slip, ω is the angular velocity, $C_x{}^V,\ C_x{}^\beta,\ C_x{}^\alpha,\ C_x{}^\omega,\ C_y(\alpha),\ C_y{}^\omega,\ C_y{}^\beta,\ C_y{}^V,\ m_z(\alpha),\ m_z{}^\omega,\ m_z{}^\beta$ are the hydrodynamic coefficients, ρ_z is the dimensionless moment of inertia about the z axis, and $k_1,\ k_2,\ k_3$ are the coefficients of the additional apparent masses.

The coefficients $C_y(\alpha)$ and $m_z(\alpha)$ can be approximated with fair accuracy by the formulas

$$m_z(\alpha) = m_z{}^\alpha \alpha + m_1 \alpha^3,\ C_y(\alpha) = C_y{}^\alpha \alpha + C_1{}^\alpha \alpha^3$$

There are three singular points in the region selected for the change of hydrodynamic coefficients for equations (3.2). They are: the unsteady origin of coordinates, and two stable points symmetrical about the latter point.

Henceforth the variables V, α, ω, β will be denoted by $x_i\ (i = 1, \ldots, 4)$ respectively. Using the probability approach /1/, we obtained that when $t \in [t_0, T]$ the set $H_{c_0}^{V(x,a)}$ becomes the set $H_{c_0/\eta(t)}^{V(x,a)} = \{x \mid V(x, a) \leqslant c_0/\eta(t)\}$. The inequality

$$(\forall t \in F)(\forall x \in \vartheta\, (F_{c(t)}^{V(x,a)}) : c(t_0) = c_0)\ \frac{d}{dt}(V(x, a)\, \eta(t)) \leqslant 0$$

is then satisfied.

Optimization of the measure of the set $H_{c/\eta(T)}^{V(x,a)}$ was carried out for particular numerical values of the hydrodynamic coefficients of (3.2). The following results were obtained.

When the function $V(x, a)$ was selected in the form

$$V(x, a) = \sum_{i=1}^{4} a_i x_i^2 \qquad (3.3)$$

and values of a_i were taken from the intervals $a_1 \in [0, 1, 6],\ a_2 = 1,\ a_3 = [0, 1, 1],\ a_4 \in [1, 3]$ and $c_0 = 0,\ 12$, the optimum function $\eta(t)$ obtained on a computer with an accuracy 0.999, is given in Fig.4.

Compared with the choice of the Lyapunov-type function of the form

$$V(x, a) = \sum_{i=1}^{4} x_i^2 \qquad (3.4)$$

and its corresponding function $\eta_1(t)$ (Fig.4) the change of measure of the set $H_{c/\eta(T)}^{V(x,a)}$ appears in Fig.5, where the dependence of the ratio $\lambda = \mathrm{mes}_{R^4} H^V_{c/\eta(t)} / \mathrm{mes}_{R^4} H^V_{c/\eta_1(t)}$ on t is shown.

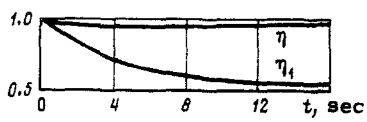

Fig.4

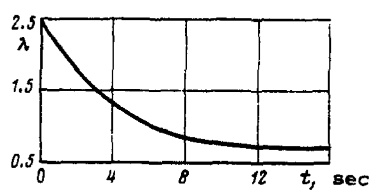

Fig.5

The curve in Fig.5 shows that the measure of set $H^V_{c/\eta(T)}$ for function (3.3) has diminished approximately threefold in comparison with (3.4).

We can similarly derive the function $\eta(t)$ satisfying the conditions of the theorems formulated above and the condition

$$(\forall c_1)(\forall t \in F)(\forall x \in H^V_{c_0} \setminus H^V_{c_1})\ \frac{d}{dt}(V(x, a)\, \eta(t)) \leqslant 0$$

Note that the determination of the stability region of the surface ψ and β when solving various applied problems provides the opportunity to analyse the evolution of sets, i.e. to consider the following problems: 1) to evaluate for a given set G_1 the set G_0, where the integral curves orignate that do not appear from G_1 in a finite time T; for a given set G_0 find the set G_1 in which the integral curves, that begin in G_0, remain and 3) to obtain analytical estimates of the attraction regions.

The solution of the second problem appears in the second example.

REFERENCES

1. VERETENNIKOV V.G. and ZAITSEV V.V., The necessary and sufficient conditions of stability in-the-large, PMM, Vol.46, No.5, 1982.
2. RIESZ F. and SZÖKEFALVI-NAGY B., Lectures on Functional Analysis. Moscow, Mir, 1979.
3. RUSH N., ABETS and LALOI M., The direct Lyapunov Method in the Theory of Stability. Moscow, Mir, 1980.
4. BHATIA N.P. and SZEGO G.P., Dynamical Systems: the Stability Theory and Applications. Berlin-Heidelberg. N.Y., Springer, 1967.
5. KRASOVSKII N.N., Some Problems of the Stability of Motion. Moscow, Fizmatgiz, 1959.
6. MATROSOV V.M., On the stability of motion. PMM, Vol.26, No.5, 1962.
7. ROCKAFELLAR R.T., Convex Analysis. Moscow, Mir, 1973.
8. BARBASHIN E.A., Introduction to the Theory of Stability. Moscow, Nauka, 1967.
9. YOSHIZAWA T., Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions. N.Y. Springer, 1975.
10. ANAPOLSKII L.YU., IRTEGOV V.D. and MATROSOV V.M., Methods of constructing Lyapunov functions. In: Achievements in Science and Technology. General Mechanics. Vol.2, Moscow, VINITI, 1975.
11. RUMYANTSEV V.V., The method of Lyapunov functions in the theory of the stability of motion. In: 50 Years of Mechanics in USSR, Vol.1, Moscow, Nauka, 1968.
12. GRUJIC L.T., Novel development of Lyapunov stability of motion. Inernat. J. Control. Vol. 22, No.4, 1975.
13. GELIG A.KH., LEONOV G.A. and YAKUBOVICH V.A., Stability of Non-linear Systems with Non-unique Equilibrium State. Moscow, Nauka, 1978.
14. ZUBOV V.I., Methods of A.M. Lyapunov and Their Application. Leningrad, Izd. LGU, 1957.
15. LA SALLE J.P., The stability theory of ordinary differential equations. J. Differ Equat., Vol.4, No.1, 1968.
16. SHESTAKOV A.A., Signs of stability of sets relative to non-autonomous differential system. Differents. Uravneniya, Vol.13, No.6, 1977.
17. MALYSHEV YU.V., Stability of sets for non-autonomous systems. Differents. Uravneniya, Vol. 16, No.5, 1980.
18. SHESTAKOV A.A. and MERENKOV YU.N., On localizing the limiting set in a non-autonomous differential system using Lyapunov functions. Differents. Uravneniya, Vol.17, No.11, 1981.
19. NEMYTSKII V.V., Topological classification of singular points and the Lyapunov generalized functions. Differents. Uravneniya, Vol.3, No.3, 1967.
20. MERKIN D.R., Introduction to the Theory of the Stability of Motion. Moscow, Nauka, 1971.
21. ANDREYEV A.S., On asymptotic stability and instability of non-autonomous systems. PMM, Vol. 43, No.5, 1979.
22. KORDUNYANU K., Application of differential inequalities in the theory of stability., Ann. Stiint Univ. Iasi. Sec. 1, Vol.6, No.1, 1960.
23. PANTOV E.N., MAKHIN N.N. and SHEREMETOV B.B., Basic Theory of the Motion of Submarines. Leningrad, Sudostroenie, 1973.

Translated by J.J.D.